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State Estimation and Stabilization of Continuous-Discrete Systems with Uncertain Disturbances

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Abstract—A nonlinear continuous-discrete system subjected to bounded exogenous disturbances is considered. The method of matrix comparison systems and the technique of differential-difference linear matrix inequalities are used to solve the following problems: state estimation via a bounding ellipsoid and the attenuation of initial deviations and uncertain disturbances via a state-feedback loop with discrete measurements. A discrete control design method is proposed to attenuate, on a finite horizon, the initial deviations and uncertain disturbances bounded by the L_∞ norm.

Keywords: system with continuous and discrete subsystems, Lipschitz nonlinearities, uncertain disturbances, state estimation, discrete control, differential-difference linear matrix inequalities

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1. INTRODUCTION

Ordinary differential-difference equations have long been a topic of research [1–6] in connection with studying partial differential equations and delay equations. As noted in the literature (see [3, 7] and references therein), many complex engineering systems described by partial differential or delay equations can be approximated by or written as simpler models in the form of differential-difference equations. Interrelated differential-difference equations are also an important class because they represent models of systems with digital computation [3] or discrete data received over a communication network [1] to control continuous-time plants. In addition, they arise in the modeling of production processes, road traffic, and biological processes in nature that evolve continuously in time and are controlled by discrete events changing their parameters or state [3].

Differential-difference equations belong to the class of hybrid systems since they describe heterogeneous interacting processes occurring in continuous and discrete time. In recent years, much attention has been paid to the stability analysis of systems with sampling (sampled-data systems), which can also be represented by differential-difference equations; see the overview [8] and references therein. This range of problems also includes control of a continuous-time system with a discrete controller or control of a networked system in which many components have continuous-time dynamics while others evolve only at discrete time instants. The existing approaches to the stability analysis and design of sampled-data systems based on the technique of linear matrix inequalities (LMIs) were considered in [8]. As underlined by the authors, despite significant advances in the field, the problems of obtaining constructive stability analysis methods remain open even in the case of linear systems.

When designing discrete controllers for continuous-time systems, researchers strive to ensure stability [14, 15] or optimal performance in terms of the H_2 or H_∞ criteria [16–18]. For the class of linear continuous-time systems with aperiodic sampling, a method for constructing a stabilizing dynamic output-feedback controller was proposed in [14]. An observer-based discrete robust controller was designed from stability conditions established using a vector Lyapunov function for 2D systems [15]. Bellman's optimality principle was adopted to obtain dynamic full-order H_2- and H_∞-optimal output-feedback controllers with periodic data sampling for linear continuous time-invariant systems [18]. In particular, stability conditions and performance indices for linear systems with discrete controllers were formulated in terms of time-dependent LMIs, which can be solved numerically. Continuous-time systems with Lipschitz nonlinearities, uncertain disturbances, and discrete control were considered in [9]. State estimation methods with bounding ellipsoids for processes with initial data from a given ellipsoid were proposed. Boundedness conditions on a finite horizon were derived in the form of solvability of a constrained optimization problem with differential-difference LMIs. With the piecewise linear approximation of the solution of differential LMIs, the problems of state estimation and discrete control design were reduced to a set of constrained optimization problems with differential LMIs, and semidefinite programming methods were used to solve them numerically. The approach developed in [9] was applied in [10–13] for state estimation and control design of nonlinear systems with discrete measurements.

In all these works on stability analysis, performance indices, and discrete control design for continuous-time systems, the discrete part determining the control change has a partial form and is described, as a rule, by a linear difference equation with constant coefficients. Exogenous disturbances are neglected, and the impact of the discrete subsystem on the continuous one is implemented only through the control variable. A new stability condition for coupled differential-functional equations was proposed in [7]. A Lyapunov–Krasovskii functional was constructed for the special case of linear differential equations with extra interaction (not only through the control variable). The stability condition was represented in terms of LMIs, convenient for numerical computation. As emphasized therein, the problem has other difficulties to overcome: stability analysis and control design require using mixed continuous- and discrete-time methods due to the hybrid nature of the entire system. In addition, the controller is usually obtained as a discontinuous function at the sampling points of the discrete subsystem. Finally, note alternative approaches to the stabilization of discrete-continuous systems that have appeared recently [19–23].

In this paper, we present state estimation, in the form of a bounding ellipsoid, as well as discrete control design methods for the class of continuous-discrete systems with Lipschitz nonlinearities and uncertain norm-bounded disturbances. Originally proposed in [24] and further developed in [25, 26], the approach involving a quadratic Lyapunov function with time-varying coefficients and differential LMIs is applied to estimate the state and design a discrete controller for the above class of systems. As a result, both problems (state estimation on a finite horizon and discrete control design) are reduced to a set of constrained optimization problems with LMIs obtained by the piecewise linear approximation of the solution of the differential LMIs [27]. The results are illustrated by an example.

2. CONTINUOUS-DISCRETE SYSTEM

Consider a system consisting of continuous and discrete subsystems interacting with each other:

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + \Phi_1\varphi_1(t, x_1(t)) + A_{12}x_2(t_k) + D_1w_1(t), \\ x_2(t_{k+1}) &= A_2x_2(t_k) + \Phi_2\varphi_2(t_k, x_2(t_k)) + A_{21}x_1(t_k) + D_2w_2(t_k), \end{aligned} \tag{1}$$

where $x_1 \in \mathrm{R}^{n_1}$ and $x_2 \in R^{n_2}$ are the state vectors of the continuous and discrete subsystems, respectively, $x_2(t) = x_2(t_k)$ for $t \in [t_k, t_{k+1})$, $t_k \in \Theta = \{t_k,\ t_k = t_{k-1} + h,\ k = 1, \ldots, N\}$; h is a discrete time step (sampling period); $w_1(t) \in \mathrm{W}_1 \subset \mathrm{R}^{r_1}$ and $w_2(t) \in \mathrm{W}_2 \subset \mathrm{R}^{r_2}$ are the vectors of uncertain exogenous disturbances; $A_i \in \mathrm{R}^{n_i \times n_i}$, $A_{12} \in \mathrm{R}^{n_1 \times n_2}$, $A_{21} \in \mathrm{R}^{n_2 \times n_1}$, $D_i \in \mathrm{R}^{n_i \times r_i}$, and $\Phi_i \in \mathrm{R}^{n_i \times q_i}$ are known matrices with constant elements; $t \in T$, $T = [t_0, t_N]$, where t_0 and t_N are initial and final time instants.

The nonlinear vector functions $\varphi_i(t, x_i)$ are continuous, bounded, and satisfy the condition

$$\|\varphi_i(t, x_i)\|^2 \leqslant \mu_i \|C_{fi} x_i\|^2 \quad \forall t \in T, \quad x_i \in \mathrm{R}^{n_i}, i = 1, 2, \tag{2}$$

where $C_{fi} \in \mathrm{R}^{q_i \times n_i}$ are known matrices with constant elements. From this point onwards, $\|\cdot\|$ denotes the Euclidean vector norm and $\mu_i > 0$, $i = 1, 2$, are known constants.

The uncertain disturbances are continuous bounded functions at each time instant:

$$W_i = \left\{ w_i(t) \in \mathrm{R}^{r_i} : \|w_i(t)\|^2 \leqslant 1 \ \ \forall t \in T \right\}, \quad i = 1, 2. \tag{3}$$

3. STATE ESTIMATION

Assume that at the initial time instant, the states $x_i(t_0) = x_{i0}$ of the continuous and discrete subsystems belong to given ellipsoids

$$E(Q_{i0}) = \left\{ x_i \in \mathrm{R}^{n_i} : x_i^{\mathrm{T}} Q_{i0}^{-1} x_i \leqslant 1 \right\}, \tag{4}$$

where $Q_{i0} (i = 1, 2)$ are given positive definite matrices.

Let $x(t) = (x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t) = x_2^{\mathrm{T}}(t_k))^{\mathrm{T}}$ denote the state vector of the continuous-discrete system.

We introduce the following notion.

Definition 1. An ellipsoid $E(Q_x(t)) = \{x \in \mathrm{R}^n : x^{\mathrm{T}} Q_x^{-1}(t) x \leqslant 1\}$ is said to be bounding for the processes $x(t, t_0, x_0)$ of the continuous-discrete system (1) evolving from an initial ellipsoid $E(Q_{x0} = \mathrm{diag}(Q_{10}, Q_{20}))$ if $x(t, t_0, x_0) \in E(Q_x(t))$ for all $t \in [t_0, t_N]$, all nonlinearities from (2), and all disturbances from (3).

Problem 1. On the finite horizon $[t_0, t_N]$ under consideration, it is required to find the matrix $Q_x(t)$ of a bounding ellipsoid $E(Q_x(t))$ for the set of processes of the original system (1) with initial data from (4), all nonlinearities from (2), and all disturbances from (3).

Note that a bounding ellipsoid will be an upper estimate of the reachability domain of the system.

To solve this problem, we introduce an augmented state vector of the form $z(t) = (x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t) = x_2^{\mathrm{T}}(t_k), x_3^{\mathrm{T}}(t) = x_1^{\mathrm{T}}(t_k))^{\mathrm{T}}$: on continuity intervals $[t_k, t_k + h)$, its components $x_1(t)$ change according to equation (1) while $x_2(t)$ and $x_3(t)$ remain invariable. At discrete time instants $t_k \in \Theta$, the components $x_2(t)$, $x_3(t)$ change jump-wise according to equation (2) and the relation $x_3(t_k) = x_1(t_k)$. Now the original continuous-discrete system (1) can be represented as the continuous system (5) with the impulses (6):

$$\dot{z}(t) = A_{z1} z + \Phi_{z1} \varphi(t, z(t)) + D_{z1} w(t), \quad t \neq t_k, \tag{5}$$

$$z(t_{k+1}) = A_{z2} z(t_k) + \Phi_{z2} \varphi(t_k, z(t_k)) + D_{z2} w(t_k), \quad t_k \in \Theta, \tag{6}$$

where

$$w(t) = \left(w_1^{\mathrm{T}}(t), w_2^{\mathrm{T}}(t) \right)^{\mathrm{T}}, \quad A_{z1} = \begin{bmatrix} A_1 & 0 & A_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{z1} = \begin{bmatrix} D_1 \\ 0 \\ 0 \end{bmatrix},$$

$$\Phi_{z1} = \begin{bmatrix} \Phi_1 \\ 0 \\ 0 \end{bmatrix}, \quad A_{z2} = \begin{bmatrix} I & 0 & 0 \\ 0 & A_2 & A_{21} \\ I & 0 & 0 \end{bmatrix}, \quad D_{z2} = \begin{bmatrix} 0 \\ D_2 \\ 0 \end{bmatrix}, \quad \Phi_{z2} = \begin{bmatrix} 0 \\ \Phi_2 \\ 0 \end{bmatrix}.$$

In addition,

$$z(t_0) = z_0 = (x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}, x_{10}^{\mathrm{T}})^{\mathrm{T}} \in E(Q_0), \quad Q_0 = \mathrm{diag}(Q_{10}, Q_{20}, Q_{10}),$$
$$x_1(t) = C_1 z(t), \quad x_2(t) = C_2 z(t),$$
$$C_1 = (\ I_{n_1} \quad 0_{n_1\times(n_2+n_1)}\ ), \quad C_2 = (\ 0_{n_2\times n_1} \quad I_{n_2} \quad 0_{n_2\times n_1}\ ),$$

where I_{n_i} $(i = 1, 2)$ is an identity matrix of dimensions $(n_i \times n_i)$.

First, we will find the matrix $Q(t)$ of a bounding ellipsoid for the states of the augmented system (5), (6). Then the matrix $Q_x(t)$ is given by $Q_x = C_{12}Q(t)C_{12}^{\mathrm{T}}$, where $C_{12} = (C_1^{\mathrm{T}}, C_2^{\mathrm{T}})^{\mathrm{T}}$.

On the continuity intervals $[t_k, t_{k+1})$ $(k = 0, 1, \ldots, N-1)$, the matrix $Q(t)$ of a bounding ellipsoid $E(Q(t))$ for the states of the augmented system will be obtained using Theorems 1 and 2 of [25], provided below in a unified formulation under the constraint imposed on nonlinearities.

Theorem 1 (see [25]). *An ellipsoid $E(Q(t))$ is bounding for the trajectories of system* (5) *evolving from an initial ellipsoid $E(Q_k)$ if one of the following conditions holds for $t \in [t_k, t_{k+1})$, $\beta_1 > 0$, and $\alpha_1 > 0$:*

1) *There exists a solution $Q(t) = Q(t, t_k, Q_k) > 0$ of the differential matrix equation*

$$dQ(t)/dt = A_{z1}Q + QA_{z1}^{\mathrm{T}} + \alpha_1 Q + \frac{1}{\alpha_1}D_{z1}D_{z1}^{\mathrm{T}} + \beta_1\Phi_{z1}\Phi_{z1}^{\mathrm{T}} + \frac{\mu_1}{\beta_1}QC_1^{\mathrm{T}}C_{f1}^{\mathrm{T}}C_{f1}C_1Q. \tag{7}$$

2) *For a fixed $\alpha_1 > 0$, there exists a solution $Q(t) = Q(t, t_k, Q_k) > 0$ of the differential LMI*

$$\begin{bmatrix} -dQ(t)/dt + A_{z1}Q + QA_{z1}^{\mathrm{T}} + \alpha_1 Q + \beta_1\Phi_{z1}\Phi_{z1}^{\mathrm{T}} & D_{z1} & QC_1^{\mathrm{T}}C_{f1}^{\mathrm{T}} \\ D_{z1}^{\mathrm{T}} & -\alpha_1 I & 0 \\ C_{f1}C_1Q & 0 & -\dfrac{\beta_1}{\mu_1}I \end{bmatrix} \leqslant 0. \tag{8}$$

Here, $Q(t_0) = Q_0$.

The proof of Theorem 1 was provided in [25].

Also, the following analytical expressions for the free parameters β_1 and α_1 were derived in [9]:

$$\beta_1(Q(t)) = +\sqrt{\frac{\mu_1 \mathrm{trace}(Q(t)C_1^{\mathrm{T}}C_{f1}^{\mathrm{T}}C_{f1}C_1Q(t))}{\mathrm{trace}(\Phi_{z1}\Phi_{z1}^{\mathrm{T}})}},$$

$$\alpha_1(Q(t)) = \sqrt{\frac{\mathrm{trace}(D_{z1}D_{z1}^{\mathrm{T}})}{\mathrm{trace}(Q(t))}}.$$

They yield locally optimal estimates by the trace criterion of the matrix $Q(t) = Q(t, t_k, Q_k)$, which is the sum of the semi-axis of the bounding ellipsoid.

For each fixed α_1 and variable β_1, the procedure for calculating the matrix $Q(t)$ of a bounding ellipsoid for the states of (5) using the differential LMI (8) is reduced, via sampling, to a set of optimization problems $\mathrm{trace}(Q(t_k)) \to \min_{Q(t_k),\beta_1(t_k)}$ with LMI constraints. This procedure is presented at the end of the current section.

At time instants t_{k+1}, $k = 0, \ldots, N-1$, the behavior of the system is described by the difference equation (6). In this case, Theorem 1 of [26] will be used to calculate the matrix $Q(t_{k+1})$ of a bounding ellipsoid for the states of the augmented system at discrete time instants t_{k+1}. Here, we present it as applied to the difference equation (6).

Theorem 2 (see [26]). *An ellipsoid $E(Q(t_{k+1}))$ is bounding for the system states at a time instant t_{k+1} under $z(t_k) \in E(Q(t_k))$ if one of the following conditions holds for $0 < \alpha_2 < 1$:*

1) *There exists a solution $Q(t_{k+1}) > 0$ of the difference matrix equation*

$$Q(t_{k+1}) = A_{z2}Q(t_k)\left[\alpha_2 Q(t_k) - \frac{\mu_2}{\beta_2}Q(t_k)C_2^{\mathrm{T}}C_{f2}^{\mathrm{T}}C_{f2}C_2Q(t_k)\right]^{-1}Q(t_k)A_{z2}^{\mathrm{T}} + \frac{1}{1-\alpha_2}D_{z2}D_{z2}^{\mathrm{T}} + \beta_2\Phi_{z2}\Phi_{z2}^{\mathrm{T}}. \tag{9}$$

2) *There exists a solution $Q(t_{k+1}) > 0$ of the difference LMI*

$$\begin{bmatrix} Q(t_{k+1}) - \beta_2\Phi_{z2}\Phi_{z2}^{\mathrm{T}} & A_{z2}Q(t_k) & D_{z2} & 0 \\ Q(t_k)A_{z2}^{\mathrm{T}} & \alpha_2 Q(t_k) & 0 & Q(t_k)C_2^{\mathrm{T}}C_{f2}^{\mathrm{T}} \\ D_{z2}^{\mathrm{T}} & 0 & (1-\alpha_2)I & 0 \\ 0 & C_{f2}C_2Q(t_k) & 0 & \dfrac{\beta_2}{\mu_2}I \end{bmatrix} \geqslant 0. \tag{10}$$

The proof of this theorem for the discrete-time system (6) with nonlinearities from (2) and uncertain disturbances from (3) was presented in [26].

Assume that the discrete time varies with a constant step $t_{k+1} - t_k = h = \text{const}$, $k = 1, \ldots, N$. Note that the case of varying the discrete time with a non-fixed known step h_k can be considered by analogy. The case where all steps h_k are unknown and vary within a given range, $h_k \in [h_{\min}, h_{\max}]$, requires separate analysis.

Well, we have the following result for the continuous-discrete system with a constant step of the discrete time.

Theorem 3. *An ellipsoid $E(Q(t))$, where $Q(t) = Q(t, t_0, Q_0)$ is a solution of the matrix system of differential-difference equations* (7), (9) *or the constrained optimization problem* $\mathrm{trace}(Q(t, t_0, Q_0)) \to \min$ *subject to the differential LMI* (8) *and the difference LMI* (10)*, is bounding for the states of system* (5), (6) *and the states of the original continuous-discrete system* (1) *for all nonlinearities from* (2) *and all disturbances from* (3).

The proof is based on Theorem 1, sequentially applied on the continuity intervals $[t_k, t_{k+1})$ $(k = 0, 1, \ldots, N-1)$, to obtain the matrix $Q(t, t_k, Q(t_k)) > 0$ of a bounding ellipsoid for the state $z(t, t_k, z(t_k))$ of system (5), (6) with initial data from the ellipsoid with the matrix $Q(t_k)$ for all nonlinearities from (2) and all disturbances from (3), and Theorem 2, sequentially applied at the points t_{k+1}, $(k = 0, 1, \ldots, N-1)$, to obtain the matrix $Q(t_{k+1}) > 0$ of a bounding ellipsoid for the state $z(t_{k+1}, t_k, z(t_k))$ of system (5), (6) at a time instant t_{k+1} under the condition $z(t_k) \in E(Q(t_k))$.

Thus, under a periodic change of discrete time, the state of system (5), (6) with initial data from the ellipsoid $E(Q_0)$ will be bounded by an ellipsoid with the matrix function $Q(t, t_0, Q_0)$ representing the solution of the matrix comparison system (7) with the impulses (9) or the constrained optimization problem $\mathrm{trace}(Q(t, t_0, Q_0)) \to \min$ subject to the differential LMI (8) and the difference LMI (10) for $t \in T$. In this case, the matrix $Q_x(t, t_0, Q_0)$ is given by $Q_x(t, t_0, Q_0) = C_{12}Q(t, t_0, Q_0)C_{12}^{\mathrm{T}}$, where $C_{12} = (C_1^{\mathrm{T}}, C_2^{\mathrm{T}})^{\mathrm{T}}$.

When numerically solving the constrained optimization problem with the differential LMI (8), sampling is performed on the time interval $[t_0, t_N]$ [27]. The derivative $dQ(t)/dt$ on the intervals $[t_k, t_{k+1})$ is supposed to be constant, $dQ(t)/dt = Z(t_k)$, where $t_k = t_0 + kh$, $k = 1, \ldots, N$, and N is the integer part of the value $(t_N - t_0)/h$. Then, for $t \in [t_k, t_{k+1})$, the matrix $Q(t)$ is given by

$$Q(t) = Q(t_k) + (t - t_k)Z(t_k), \tag{11}$$

and $Q(t_0) = Q_0$. The matrix $Q(t)$ will satisfy the inequality $Q(t) > 0$ and the differential LMI (8) for all $t \in [t_k, t_{k+1})$ iff it satisfies them at the two extreme points t_k and $t_k + h$. In other words, for each $k = 0, \ldots, N-1$, the following inequalities must hold simultaneously [27]:

$$Q(t_k) > 0, \quad Q(t_k + h) > 0, \tag{12}$$

$$\begin{bmatrix} -Z(t_k) + F(Q(t_k)) + \beta_1 \Phi_{z1} \Phi_{z1}^{\mathrm{T}} & D_{z1} & Q(t_k) C_1^{\mathrm{T}} C_{f1}^{\mathrm{T}} \\ D_{z1}^{\mathrm{T}} & -\alpha_1 I & 0 \\ C_{f1} C_1 Q(t_k) & 0 & -\dfrac{\beta_1}{\mu_1} I \end{bmatrix} \leqslant 0, \tag{13}$$

$$\begin{bmatrix} -Z(t_k) + F(Q(t_k + h)) + \beta_1 \Phi_{z1} \Phi_{z1}^{\mathrm{T}} & D_{z1} & Q(t_k + h) C_1^{\mathrm{T}} C_{f1}^{\mathrm{T}} \\ D_{z1}^{\mathrm{T}} & -\alpha_1 I & 0 \\ C_{f1} C_1 Q(t_k + h) & 0 & -\dfrac{\beta_1}{\mu_1} I \end{bmatrix} \leqslant 0, \tag{14}$$

where $Q(t_k + h) = Q(t_k) + hZ(t_k) = Q(t_{k+1} - 0)$ and $F(G) = A_{z1} G + G A_{z1}^{\mathrm{T}} + \alpha_1 G$.

After the linear approximation (11) of the solution of the differential LMI (8), the matrix $Q(t) > 0$ of a bounding ellipsoid for the system states is found by sequentially solving the set of constrained optimization problems $\operatorname{trace}(Q(t_{k+1})) \to \min_{Q(t_{k+1})>0, \beta_1(t_{k+1})>0}$ subject to the LMIs (10), (12)–(14) for $k = 0, \ldots, N-1$. At the first iteration ($k = 0$), for a matrix $Q(t_0) = Q_0$, the matrices $Q(t_0 + h)$ and $Q(t_1)$ with minimal trace are calculated by solving the above optimization problem with the LMI constraints. These matrices determine the matrix of a bounding ellipsoid for the states of system (5), (6) on the interval $[t_0, t_1]$. Then, for $k = 1, \ldots, N-1$, the matrices $Q(t_k + h)$ and $Q(t_{k+1})$ are calculated from the matrix $Q(t_k)$. These matrices determine the matrix of a bounding ellipsoid for the states of system (5), (6) on the subsequent intervals $[t_k, t_{k+1}]$.

Note that at each iteration, semidefinite programming tools (CVX, Sedumi, Yalmip, etc.) can be used for numerical solution of optimization problems with LMI constraints.

4. DISCRETE CONTROL DESIGN TO ATTENUATE THE INITIAL DEVIATIONS AND UNCERTAIN DISTURBANCES OF THE CONTINUOUS-DISCRETE SYSTEM

Consider the continuous-discrete system (1) with discrete control:

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t_k) + \Phi_1 \varphi_1(t, x_1(t)) + A_{12} x_2(t_k) + D_1 w_1(t), \\ x_2(t_{k+1}) &= A_2 x_2(t_k) + B_2 u(t_k) + \Phi_2 \varphi_2(t_k, x_2(t_k)) + A_{21} x_1(t_k) + D_2 w_2(t_k). \end{aligned} \tag{15}$$

Problem 2. It is required to find a state-feedback controller with available discrete measurements at time instants t_k, $k = 0, 1, \ldots, N-1$, that stabilizes the closed-loop system and attenuates initial deviations and exogenous disturbances in the sense of the minimal bounding ellipsoid for the states of the continuous-discrete system.

For each subsystem, the state-feedback controller is based on the state of the entire system at time instants t_k :

$$u_i(t) = K_{i1}(t_k) x_1(t_k) + K_{i2}(t_k) x_2(t_k), \quad t \in [t_k, t_{k+1}), \quad i = 1, 2, \tag{16}$$

where $K_{ij}(t_k) - (m_i \times n_j)$, $i = 1, 2$, $j = 1, 2$, are the gain matrices of the discrete controllers and $k = 0, 1, \ldots, N-1$.

The controller $u_i(t)$ must satisfy the constraint

$$u_i(t) \in \left\{ u_i : u_i^{\mathrm{T}} U_i^{-1} u_i \leqslant 1 \right\}, \quad t \in T, \tag{17}$$

where U_i is a given symmetric positive definite matrix of dimensions $(m_i \times m_i)$ $(i = 1, 2)$.

We write system (15) with the discrete controller (16) as

$$\dot{z}(t) = A_{z1}z + \Phi_{z1}\varphi_1(t, z(t)) + D_{z1}w(t), \quad t \neq t_k, \tag{18}$$

$$z(t_{k+1}) = A_{z2}z(t_k) + \Phi_{z2}\varphi_2(t_k, z(t_k)) + D_{z2}w(t_k), \quad t_k \in \Theta, \tag{19}$$

where $z(t) = (x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t_k), x_1^{\mathrm{T}}(t_k), u^{\mathrm{T}}(t_k))^{\mathrm{T}}$ is the state vector of dimension $n = n_1 + n_2 + n_1 + m$, $u(t_k) = (u_1^{\mathrm{T}}(t_k),\ u_2^{\mathrm{T}}(t_k))^{\mathrm{T}}$ is the control vector of dimension $m = m_1 + m_2$, $t \in [t_k, t_{k+1})$, and

$$A_{z1} = \begin{bmatrix} A_1 & A_{12} & 0 & B_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{z2} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & A_2 & A_{21} & B_2 \\ I & 0 & 0 & 0 \\ 0 & K_{12} & K_{11} & 0 \\ 0 & K_{22} & K_{21} & 0 \end{bmatrix}, \quad \Phi_{z1} = \begin{bmatrix} \Phi_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{z1} = \begin{bmatrix} D_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Phi_{z2} = \begin{bmatrix} 0 \\ \Phi_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{z2} = \begin{bmatrix} 0 \\ D_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_{12}(t_k) & K_{11}(t_k) \\ K_{22}(t_k) & K_{21}(t_k) \end{bmatrix}.$$

In view of the numerical solution method for the differential LMI (8) (Section 3), the control design problem for system (18), (19) reduces to a constrained optimization problem with difference LMIs. In this problem, the criterion is the trace of the matrix $Q(t_k)$, determining the size of a bounding ellipsoid $E(Q(t_k))$ for the state $z(t_k)$ for all t_k, $k = 1, \ldots, N$.

The gain matrix K of the controllers appears only in the difference equation (6). Let us represent (6) in the form

$$z(t_{k+1}) = (\bar{A}_{z2} + \bar{B}KC)z(t_k) + \Phi_{z2}\varphi(t_k, z(t_k)) + D_{z2}w(t_k), \quad t_k \in \Theta, \tag{20}$$

where $\bar{A}_{z2} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & A_2 & A_{21} & B_2 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix}$, $C = \begin{bmatrix} 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 \end{bmatrix}$.

The following result is true.

Theorem 4. *Assume that for some* $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ *and all* t_k, $k = 0, 1, \ldots, N-1$, *there exists a solution* $Q(t_{k+1}) > 0$, $Q(t_{k+1} - 0) = Q(t_k + h) = Q(t_k) + hZ(t_k) > 0$, $Y(t_k)$ *of the problem*

$$\operatorname{trace}[Q(t_{k+1})] \to \min$$

subject to the constraints (12)–(14), (21), *and* (22):

$$\begin{bmatrix} Q(t_{k+1}) - \beta_2\Phi_{z2}\Phi_{z2}^{\mathrm{T}} & \bar{A}_{z2}Q(t_k)C^{\mathrm{T}} + BY_k & D_{z2} & 0 \\ CQ(t_k)\bar{A}_{z2}^{\mathrm{T}} + Y_k^{\mathrm{T}}\bar{B}^{\mathrm{T}} & \alpha_2 CQ(t_k)C^{\mathrm{T}} & 0 & CQ(t_k)C_2^{\mathrm{T}}C_{f2}^{\mathrm{T}} \\ D_{z2}^{\mathrm{T}} & 0 & (1-\alpha_2)I & 0 \\ 0 & C_{f2}C_2Q(t_k)C^{\mathrm{T}} & 0 & \dfrac{\beta_2}{\mu_2}I \end{bmatrix} \geqslant 0, \tag{21}$$

$$\begin{pmatrix} U & Y_k \\ Y_k^{\mathrm{T}} & CQ(t_k)C^T \end{pmatrix} \geqslant 0, \tag{22}$$

where minimization is carried out with respect to the matrix variables $Q(t_{k+1}) \in \mathrm{R}^{n\times n}$, $Y_k \in \mathrm{R}^{m\times(n_2+n_1)}$, $Z(t_k) \in \mathrm{R}^{n\times n}$ *and the scalar variables* $\beta_1(t_k)$ *and* $\beta_2(t_k) > 0$. *Then the gain matrix of the discrete state-feedback controller stabilizing the continuous-discrete system is given by* $K(t_k) = Y_k(CQ(t_k)C^{\mathrm{T}})^{-1}$. *In this case, the stabilization errors are estimated by a bounding ellipsoid for the state vector* $z(t)$, *and the matrix* $Q(t)$ *of this ellipsoid is given by* (11).

Note that the matrix $CQ(t_k)C^{\mathrm{T}}$ is positive definite since it represents the middle block of dimensions $(n_2 + n_1) \times (n_2 + n_1)$ in the matrix $Q(t_k)$.

5. AN ILLUSTRATIVE EXAMPLE

Consider the continuous-discrete system (15) with nonlinearities $\varphi_i(x_i)$, $i = 1, 2$, from (2), disturbances $w_i(t)$, $i = 1, 2$, from (3), and the parameters

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -5.04 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1.36 & 0 \end{bmatrix},$$

$$\Phi_1 = \begin{bmatrix} 0 \\ -1.87 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ -0.95 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 1 & 1.25 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$\mu_1 = 1, \quad \mu_2 = 0.5, \quad C_{f1} = C_{f2} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

It is required to design a discrete controller ensuring finite stabilization with the maximum sampling period h.

Based on Theorem 4 and the numerical solution of the corresponding set of optimization problems with LMI constraints, $q_1 = 0.03$, $q_2 = 0.79$, $Q_0 = \mathrm{diag}\{\ 0.5\ \ 0.5\ \ 0.5\ \ 0.5\ \ 0.5\ \ 0.5\ \ 0\ \ 0\ \}$, and a sampling period of $h = 2.5$ s, we obtained the controller (16) with variable gains $K(t_k)$. Figure 1 shows these gains for the continuous and discrete subsystems on the horizon [0, 60] s.

The continuous-discrete system was simulated under the particular nonlinearities $\varphi_1(x_1) = \sin(C_{f1}x_1)$ and $\varphi_2(x_2) = 0.7071\,|C_{f2}x_2|$, which satisfy condition (2). The simulation results for

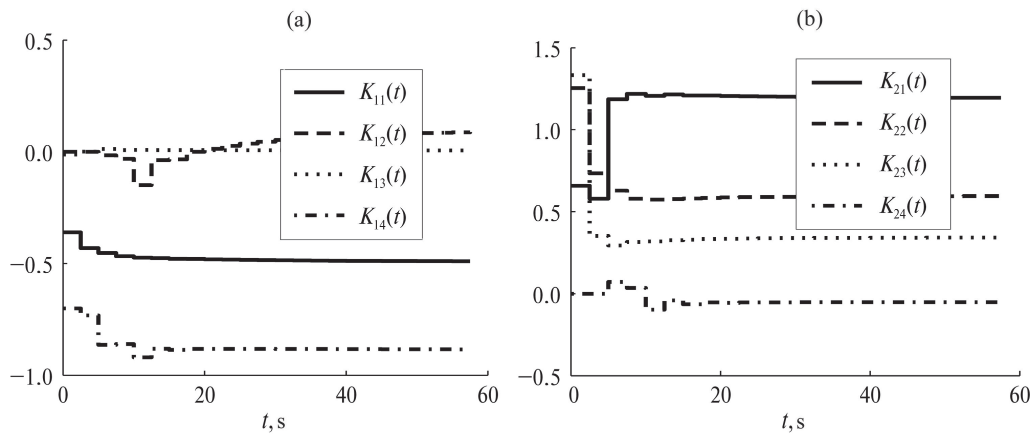

Fig. 1. The variable gains of the controller: (a) the continuous subsystem and (b) the discrete subsystem.

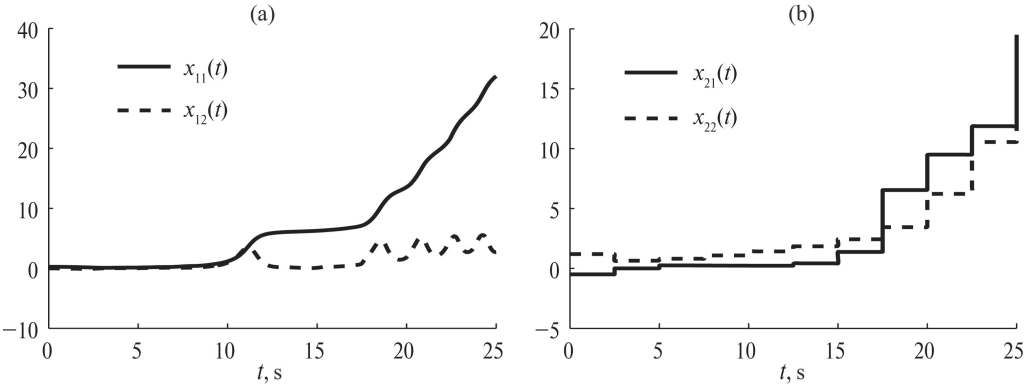

Fig. 2. System state without the controller and disturbances: (a) the continuous subsystem and (b) the discrete subsystem.

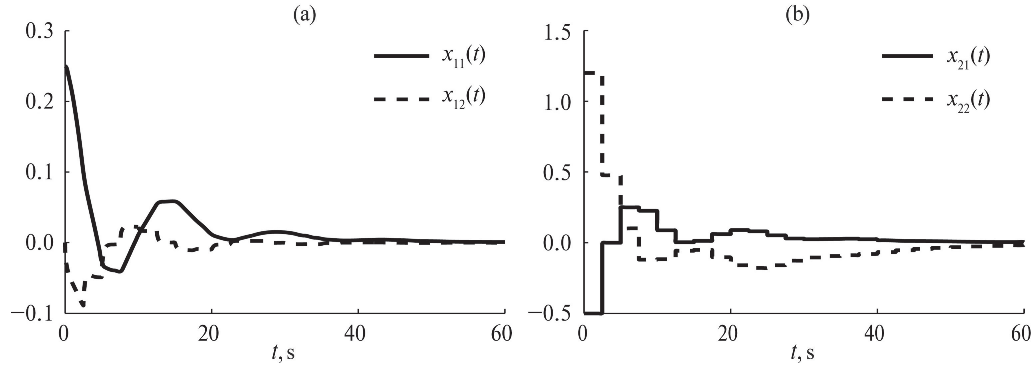

Fig. 3. System state without the disturbances: (a) the continuous subsystem and (b) the discrete subsystem.

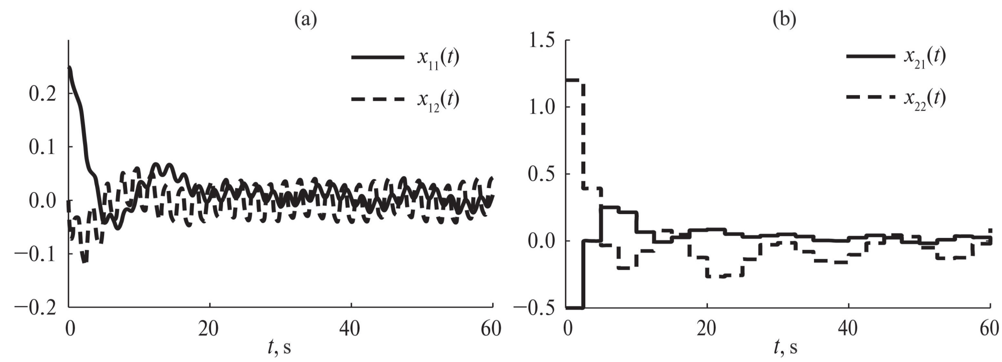

Fig. 4. System state with the disturbances: (a) the continuous subsystem and (b) the discrete subsystem.

the system without the controller and disturbances with a sampling period of $h = 2.5$ s are presented in Fig. 2. According to the graphs, the system without the controller is unstable.

The continuous-discrete system with the resulting controller was simulated under different initial conditions. Figure 3 shows the transients in the continuous and discrete subsystems with this controller under the initial conditions $x_0 = \begin{bmatrix} 0.25 & 0 & -0.5 & 1,2 \end{bmatrix}^{\mathrm{T}}$ without disturbances. Next, Fig. 4 demonstrates the transients in the continuous and discrete subsystems with this controller under the same initial conditions and the disturbances given by the functions $w_1(t) = \sin(2\cos(3t))/5$ and $w_2(t_k) = \sin(k)/10$. In this case, the control signals of the continuous and discrete subsystems are presented in Fig. 5.

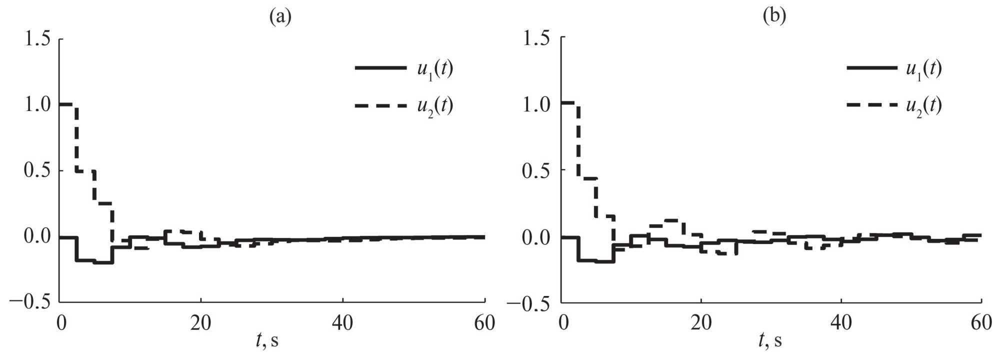

Fig. 5. The control signals of the continuous and discrete subsystems: (a) without the disturbances and (b) with the disturbances.

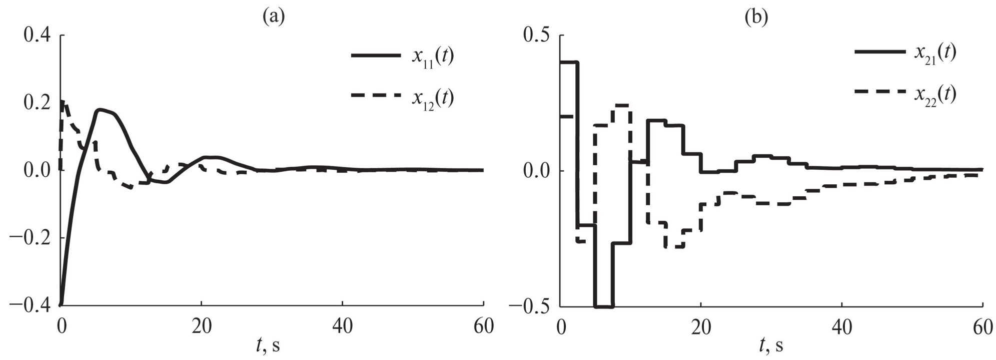

Fig. 6. System state without the disturbances and with the initial conditions $x_0 = [\ -0.4\ \ 0\ \ 0.4\ \ 0.2\ ]^{\mathrm{T}}$: (a) the continuous subsystem and (b) the discrete subsystem.

Finally, Fig. 6 shows the states of the continuous and discrete subsystems obtained by simulation under other initial conditions: $x_0 = \left[\ -0.4\ \ 0\ \ 0.4\ \ 0.2\ \right]^{\mathrm{T}}$.

According to Figs. 3–6, the discrete controller designed stabilizes the continuous-discrete system with a sampling period of $h = 2.5$ s.

6. CONCLUSIONS

This paper has been devoted to continuous-discrete systems with Lipschitz nonlinearities and uncertain disturbances. We have proposed methods for solving two problems: state estimation via bounding ellipsoids for the states of processes with initial data from a given ellipsoid and discrete control design to attenuate initial deviations and uncertain disturbances. A quadratic Lyapunov function with time-varying parameters has been used to obtain boundedness conditions on a finite horizon in the form of the solvability of a constrained optimization problem with differential-difference LMIs. With the piecewise linear approximation of the solution of the differential LMI, the problems of state estimation and discrete control design have been reduced to a set of optimization problems with LMIs, and semidefinite programming methods have been employed to solve them numerically. The results have been applied to state estimation and discrete control design for finite-horizon stabilization of a particular continuous-discrete system with uncertain disturbances.

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APPENDIX

Proof of Theorem 4. In the case of periodic discrete control ($h_k = h = \text{const} > 0$, see Section 3), state estimation via a bounding ellipsoid reduces to a set of optimization problems $\text{trace}(Q(t_{k+1})) \to \min$ subject to the LMIs (10), (12)–(14) for all $k = 0, \ldots, N-1$. In view of (20), the matrix inequality (10) here takes the form

$$\begin{bmatrix} Q(t_{k+1}) - \beta_2 \Phi_{z2}\Phi_{z2}^{\mathrm{T}} & (\bar{A}_{z2} + \bar{B}KC)Q(t_k) & D_{z2} & 0 \\ Q(t_k)(\bar{A}_{z2} + \bar{B}KC)^{\mathrm{T}} & \alpha_2 Q(t_k) & 0 & Q(t_k)C_2^{\mathrm{T}}C_{f2}^{\mathrm{T}} \\ D_{z2}^{\mathrm{T}} & 0 & (1-\alpha_2)I & 0 \\ 0 & C_{f2}C_2Q(t_k) & 0 & \dfrac{\beta_2}{\mu_2}I \end{bmatrix} \geqslant 0.$$

Then, multiplying the last inequality by the matrices $\text{diag}(I, C, I, I)$ and $\text{diag}(I, C^{\mathrm{T}}, I, I)$ on the left and right, respectively, and introducing the change of variables $Y_k = KCQ(t_k)C^{\mathrm{T}}$, we arrive at the difference LMI (21) with respect to the matrix variables $Q(t_{k+1})$ and Y_k. The control constraint (17) is ensured by the LMI (22), where $U = \text{diag}(U_1, U_2)$.

The proof of Theorem 4 is complete.

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